The Grid-Property and Product-Like Hypergraphs

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Abstract

Equivalence relations on the edge set of a hypergraph that satisfy the "grid-property" (a certain restrictive condition on diagonal-free grids that can be seen as a generalization of the more familiar "square property" on graphs) play a crucial role in the theory of Cartesian hypergraph products. In particular, every convex relation with the grid property induces a factorization w.r.t. the Cartesian product. In the class of graphs, even non-convex relations with the square property provide rich structural information on local isomorphisms, local product structures, and product structures of quotient graphs. Here, we examine the grid property in its own right. Vertex partitions derived from these equivalence classes of the edges give rise to equivalence relations on the vertex set. This in turn determine quotient graphs that have non-trivial product structures.
Keywords: grid property, quotient hypergraph, Cartesian hypergraph product

1 Introduction

It has been known for nearly half a century that every connected hypergraph has a unique prime factor decomposition with respect to Cartesian product [5]. In the class of graphs, Cartesian products are conveniently characterized in terms of the convex closure of certain relations on the edge set that satisfies the so-called "square property", $e \sim f$ if either e and f are opposite edges of a chordless square, or e and f are adjacent and do not span a chordless square. Generalizing the approach of [7], we recently showed that an analogous result holds for Cartesian products of hypergraphs [8]. More precisely, the product relation σ on the edge set of a hypergraph H, i.e., the equivalence relation on its edge sets that identifies the fibres of the prime factors of H as the connected components of the partial hypergraphs generated by a single equivalence class of σ, can be represented as the convex closure of an initial relation δ. The latter is defined locally by a generalization of the square property known as the "grid property", which we introduce formally in the section 2.

Within the class of graphs, the relation δ [10] and its generalizations have long played an important role in the context of Cartesian products and their generalizations, the Cartesian graph bundles [6, 7, 11, 12]. A recent attempt [4] to better understand the structure of equivalence relations on the edge set of a graph G that satisfies the square property, we uncovered a surprising connection to equitable partitions on the vertex set of G and a Cartesian factorization of certain quotient graphs that was observed in the context of quantum walks on graphs [1]. Here, we explore to what extent these results can be generalized to hypergraphs.

2 The Grid Property

Throughout this contribution we consider finite, connected, simple hypergraphs H. The notations largely follows the survey of hypergraph products [3]. To make this contribution self-consistent we include the basic definitions and notations in the Appendix.

We start by defining grids in hypergraphs. As in the case of cycles in graphs, which are conveniently defined as collections of edges, we regard them as collections of (hyper)edges.

Definition 1. *[8] An* $r \times s$-grid *in a hypergraph* $H = (V,E)$ *is a collection* $\mathscr{G} = \{e_1,\dots,e_s, f_1,\dots, f_r\} \subseteq E$ *of edges such that*

(i) $|e_i \cap f_j| = 1$*, and*

(ii) $e_i \cap e_{i'} = f_j \cap f_{j'} = \emptyset$*,*

for all $i,i' \in \{1,\dots,s\},\ j, j' \in \{1,\dots,r\}$*, with* $i \neq i'$*,* $j \neq j'$*. We say that* e_i *and* e_j *as well as* f_i *and* f_j *are* parallel edges *of* $\mathscr{G}$*.*

A diagonal *in* $\mathscr{G}$ *is an edge* $d \in E(H)$ *satisfying*

$$e_k \cap f_l \cap d \neq \emptyset \quad \text{and} \quad e_{k'} \cap f_{l'} \cap d \neq \emptyset$$

for $k,k' \in \{1,\dots,s\}$ *and* $l,l' \in \{1,\dots,r\}$ *with* $k \neq k'$ *and* $l \neq l'$*,*

The significance of *diagonal-free* grids is that they appear as the Cartesian product of two hyperedges. Thus, they can be seen as a natural generalization of the chordless squares that appears as products of edges in the Cartesian graph product.

In [8], the following generalization of the "square property" for equivalence relations on the edge set of simple graphs was introduced:

Definition 2. *Let* R *be an equivalence relation on the edge set* $E(H)$ *of a hypergraph* H*. We say* R *has the* grid property *if*

(S1) Any two adjacent edges e *and* f *of* H *belonging to distinct* R*-equivalence classes span exactly one diagonal free* $|e| \times |f|$*-grid* $\mathscr{G}$ *and*

(S2) Parallel edges in any diagonalfree grid $\mathscr{G}$ *of* $E(H)$ *are in the same* R*-equivalence class.*

The restriction of these statements to simple graphs recovers the definition of the *square property* used in [4]: (S1) Any two adjacent edges e and f of H belonging to distinct R-equivalence classes span exactly one square, and (S2) Non-adjacent edges in a chordless square belong to the same R-equivalent class.

In graphs, an equivalence relation with the square property is readily constructed as the transitive closure of the relation δ [2, 7, 10]. In [8] the following generalization to hypergraphs has been introduced:

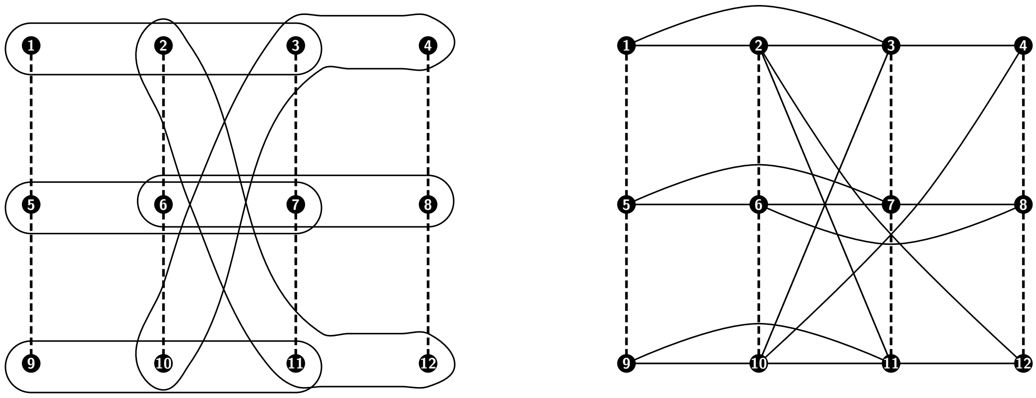

Figure 1: The l.h.s. shows a hypergraph H with an equivalence relation R on $E(H)$ that consists of two equivalence classes indicated by edges with $|e|=2$ (dashed) and edges with $|e|=3$ (solid ovals). R has the gridproperty, but $\delta \not\subseteq R$, since, e.g., $(\{2,11,12\},\{3,7\}) \in \delta$ but $(\{2,11,12\},\{3,7\}) \notin R$. The r.h.s. shows the 2-section $[H]_2$ together with the equivalence relation R_2 on $E([H]_2)$ induced by R. Its equivalence classes are depicted by dashed and solid edges, respectively. R_2 does not satisfy the square property, since edges $\{2,3\}$ and $\{2,6\}$ span more than one square.

Definition 3. *Let H be a connected hypergraph. Two edges $e, f \in E(H)$ are* in relation δ, *$e\ \delta\ f$, if one of the following conditions holds:*

(i) $e = f$

(ii) $e \cap f = \emptyset$ *and e and f are opposite edges of a four-cycle*

(iii) $e \cap f \neq \emptyset$ *and there is no $(|e| \times |f|)$-grid without diagonals containing them.*

The relation δ is reflexive and symmetric. Its transitive closure δ^* is therefore an equivalence relation. As shown in [8], δ^* has the grid property and any equivalence relation that contains δ has the grid property. Moreover, we can state the following

Proposition 1. *If R is an equivalence relation on $E(H)$ satisfying the grid property and S is a coarse equivalence relation, $R \subseteq S$, then S also has the grid property.*

Conversely, however, if a relation R on $E(H)$ satisfies the grid property, this does not necessarily imply $\delta \subseteq R$, as shown in Fig. 1. However, we can formulate more restrictive conditions in terms of the 2-section $[H]_2$ of the hypergraph H. Let R be an equivalence relation on $E(H)$. Then R induces a relation R_2 on $E([H]_2)$ by setting $e'\ R_2\ f'$ for $e', f' \in E([H]_2)$ iff there are edges $e, f \in E(H)$ with $e\,R\,f$ and $e' \subseteq e,\ f' \subseteq f$.

Proposition 2. *If R has the grid property then R_2 is an equivalence relation on $E([H]_2)$.*

Proof. Since R_2 is clearly reflexive and symmetric we only need to show that R_2 is transitive. Therefore, let $e', f', g' \in E([H]_2)$ and suppose $e'\ R_2\ f'$ and $f'\ R_2\ g'$. By construction, there are $e, f, \hat{f}, g \in E(H)$ such that $e' \subseteq e, f' \subseteq f, f' \subseteq \hat{f}, g' \subseteq g$ and $e\,R\,f$ as well as $\hat{f}\,R\,g$. Furthermore, $f' \subseteq f \cap \hat{f}$, thus, $|f \cap \hat{f}| \geq 2$, which implies $f\ R\ \hat{f}$ because R satisfies the grid-property. Since R is an equivalence relation, we can conclude $e\,R\,g$ and therefore also $e'\ R_2\ g'$. □

Definition 4. *An equivalence relation R on $E(H)$ has the* strong grid property *if R has the grid property and the induced equivalence relation R_2 on $E([H]_2)$ has the square property.*

Lemma 1. *An equivalence relation R on $E(H)$ has the strong grid property if and only if $\delta \subseteq R$.*

Proof. First, let $\delta \subseteq R$. Then R has the grid property. We have to show that R_2 on $E([H_2])$ satisfies the square property. Therefore let $e' = \{x,y\}, f' = \{y,z\} \in E([H]_2)$ be two adjacent edges such that $(e', f') \notin R_2$. Then there exists adjacent edges $e, f \in E(H)$ with $e' \subseteq e$, $f' \subseteq f$ such that $(e, f) \notin R$. Thus, e and f span a unique diagonal-free grid $\mathscr{G} = \{e, e_1, \ldots, e_k, f, f_1, \ldots, f_l\}$ with $|e| = l+1$ and $|f| = k+1$. W.l.o.g., let $e \cap f_1 = \{x\}$ and $e_1 \cap f = \{z\}$. Hence, e' and f' span a square (x,y,z,w) with $\{w\} = e_1 \cap f_1$. This square must be chordless, since for any chord d' there exists a diagonal $d \supseteq d'$ of the grid $\mathscr{G}$.

Suppose there exists another square (x,y,z,v) spanned by e' and f'. Hence, there exists $\hat{e}, \hat{f}$ in $E(H)$ such that $\{x,v\} \subseteq \hat{e}$ and $\{v,z\} \subseteq \hat{f}$. Then neither $\hat{e}$ nor $\hat{f}$ are contained in $\mathscr{G}$, since otherwise $\hat{f}$ or $\hat{e}$, respectively, would be a diagonal of $\mathscr{G}$. If $\hat{e} = \hat{f}$ holds, then $\hat{e}$ would be a diagonal of this grid. Hence, $\hat{e} \neq \hat{f}$ must hold. However, this implies $f_1 \delta \hat{f}$ as well as $e \delta \hat{f}$ and therefore $e \delta^* f$ and consequently $e\,R\,f$, a contradiction. Thus, R has the strong grid property if $\delta \subseteq R$.

Now, suppose R has the strong grid property. We have to show that for any two edges $e, f \in E(H)$ with $e \delta f$ holds $e\,R\,f$. First, suppose $e \delta f$ such that e and f are not adjacent. Thus, e and f must be opposite edges of a 4-cycle. Hence, there exists $e', f' \in E([H]_2)$ with $e' \subseteq e$ and $f' \subseteq f$ such that e' and f' are opposite edges of a square in $[H]_2$. Since R_2 has the square property, we can conclude $e'\ R_2\ f'$. That is, there exists edges $\hat{e}, \hat{f} \in E(H)$ with $e' \subseteq \hat{e}$ and $f' \subseteq \hat{f}$ such that $\hat{e} R \hat{f}$. From $|f \cap \hat{f}| \geq 2$ and $|e \cap \hat{e}| \geq 2$, we can conclude $e\,R\,\hat{e}$ and $f\,R\,\hat{f}$ and finally $e\,R\,f$. Now, let $e, f \in E(H)$ be adjacent and suppose, for contraposition, $(e, f) \notin R$. Then e and f span a unique, diagonal-free grid. The definition of δ implies $(e, f) \notin \delta$. □

Instead of δ we can also construct a less restrictive, i.e., finer, equivalence relation on $E(H)$ with the grid property.

Definition 5. *Two edges $e, f \in E(H)$ are* in relation γ *if one of the following conditions holds:*

(i) $e = f$

(ii) $e \cap f = \emptyset$ *and e and f are parallel edges in a grid $\mathscr{G}$ in H*

(iii) $e \cap f \neq \emptyset$ *and there is no diagonal-free $(|e| \times |f|)$-grid that contains e and f*

(iv) $e \cap f \neq \emptyset$ *and there is more than one diagonal-free $(|e| \times |f|)$-grid containing e and f*

By construction, γ is reflexive and symmetric. Its transitive closure γ^* is therefore an equivalence relation.

Lemma 2. *An equivalence relation R on $E(H)$ has the grid property if and only if $\gamma \subseteq R$.*

Proof. First, suppose $\gamma \subseteq R$. We have to show that R satisfies the grid property. Therefore, let $e, f \in E(H)$ be two adjacent edges such that $(e, f) \notin R$, hence, $(e, f) \notin \gamma$. Then from condition (iii) in the definition of γ, we can conclude that there exists a grid $\mathscr{G}$ in H spanned by e and f, and from condition (iv) it follows that this grid is unique.

Now, suppose R has the grid property. We have to show that $e\,R\,f$ for all $e, f \in E(H)$ with $e\,\gamma\,f$. First, suppose $e\,\gamma\,f$ and e and f are not adjacent. Thus, e and f must be parallel edges in a grid, and therefore $e\,R\,f$. Now, suppose e and f are adjacent and suppose, for

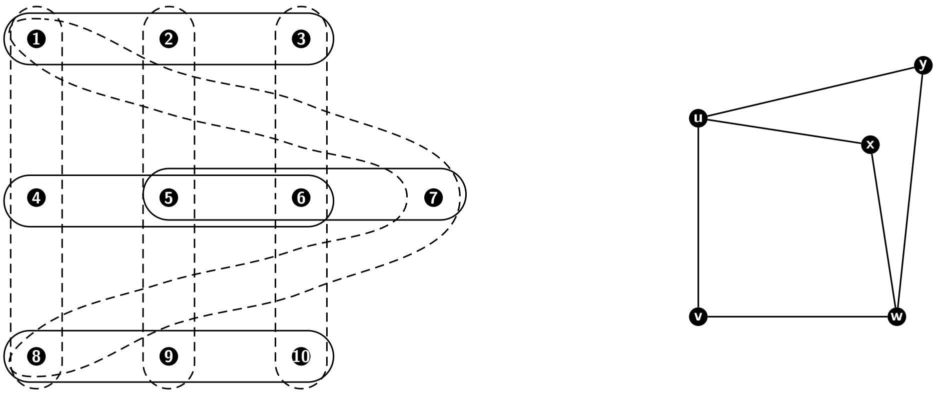

Figure 2: **LHS:** The equivalence relation γ_0 on $E(H)$, whose classes are indicated by dashed and solid ovals, resp., satisfies one of the conditions *(i)*, *(ii)*, and *(iii)* of Definition 5, but violated condition (iv) since there are two grids spanned by edges $\{1,2,3\}$ and $\{2,5,9\}$.
RHS: The edges $\{u,v\}$ and $\{v,w\}$ in the graph span two squares, $\langle\{u,v,w,x\}\rangle$ and $\langle\{u,v,w,y\}\rangle$. It holds: $\{u,v\}\gamma_0\{wy\}$, $\{w,y\}\gamma_0\{u,x\}$ and $\{u,x\}\gamma_0\{v,w\}$, hence $\{u,v\}\gamma_0^*\{v,w\}$.

contraposition, $(e,f)\notin R$. Then e and f span a unique grid without diagonals. Then, from the definition of γ we can conclude $(e,f)\notin\gamma$ since neither (iii) nor (iv) is fulfilled. □

If H is a graph, then we do not have to insist on condition (iv) in the definition of γ. To see this, consider the relation γ_0 satisfying one of the conditions *(i)*, *(ii)*, and *(iii)* in Definition 5. If e and f span more than one square in a graph, then they are guaranteed to be in the transitive closure γ_0^* of γ_0, see Fig. 2. This is not true in a hypergraph, however, as the example in Fig. 2 shows.

Corollary 1. *For any connected hypergraph H holds $\gamma_0^*\subseteq\gamma^*\subseteq\delta^*$. If H is a graph, then $\gamma_0^*=\gamma^*=\delta^*$.*

In the remainder of this section we collect several useful properties of the grid property that generalize well known results for the square property. Some of these results have been shown in previous work, others are novel to our knowledge.

We will need the following notation: Let R be an equivalence relation on $E(H)$ and let $\varphi\sqsubseteq R$ be an equivalence class of R. An edge $e\in E(H)$ is a φ-edge if $e\in\varphi$. For a given $\varphi\sqsubseteq R$ we consider the partial hypergraph $H_\varphi=(V,\varphi)$ generated by the φ-edges. The connected component of H_φ containing the vertex $x\in V(H)$ is denoted by H_φ^x. The star with center u generated by all φ-edges incident to u is denoted by $S_{H_\varphi}(u)$ or $S_\varphi(u)$ for short.

Lemma 3 ([8])**.** *Let R be an equivalence relation on $E(H)$ with the grid property. Then each vertex of H is incident to at least one edge of each R-class.*

Lemma 4 ([8])**.** *Let R be an equivalence relation on $E(H)$ with the grid property that has only two equivalence classes φ and $\overline{\varphi}$. Then $|V(H_\varphi^x)\cap V(H_{\overline{\varphi}}^y)|\geq 1$ for all $x,y\in V(H)$. If both H_φ^x and $H_{\overline{\varphi}}^y$ are convex then $|V(H_\varphi^x)\cap V(H_{\overline{\varphi}}^y)|=1$.*

Proposition 3. *Let R be an equivalence relation on $E(H)$ with the grid property, $\varphi,\psi\sqsubseteq R$, $\psi\neq\varphi$, $e\in\varphi$, and $x,y\in e$. Then $m(S_\psi(x))=m(S_\psi(y))$.*

Proof. W.l.o.g., suppose $E(S_\psi(x)) = \{f_1,\ldots,f_k\}$. The grid property implies that each f_i together with e spans a unique diagonal-free grid $\mathscr{G}_i$, $i = 1,\ldots,k$ and $\mathscr{G}_i \neq \mathscr{G}_j$ if $i \neq j$. Furthermore, for each $i = 1,\ldots,k$ there is an edge $f'_i \in \mathscr{G}_i$ such that $y \in f'_i$. Thus $\mathscr{G}_i$ is also spanned by f'_i and e. If $f'_i = f'_j$, this immediately implies $i = j$, otherwise f'_i and e would span more than one grid. Therefore, we have $m(S_\psi(x)) \leq m(S_\psi(y))$. An analogous argument established and $m(S_\psi(y)) \leq m(S_\psi(x))$, hence equality must hold. □

Lemma 5. *Let R be an equivalence relation on $E(H)$ with the strong grid property, $\varphi,\psi \sqsubseteq R$, $\psi \neq \varphi$, $e \in \varphi$, and $x,y \in e$. Then the stars generated by all ψ-edges centered in x and y, resp., are isomorphic, $S_\psi(x) \cong S_\psi(y)$.*

Proof. Let $e \in \varphi \subseteq E(H)$, $x,y \in V(H)$ such that $x,y \in e$. Let $E(S_\psi(x)) = \{f^1,\ldots,f^m\}$ and $E(S_\psi(y) = \{g^1,\ldots g^{m'}\})$. We have $m = m'$ as a consequence of Prop. 3.

By the grid property, $e \in \varphi$ and $f^i \in \psi$ span a unique grid $\mathscr{G}^i = \{f^i, f^i_1,\ldots,f^i_l, e, e^i_1,\ldots,e^i_{k_i}\}$ in H for all $i = 1,\ldots,m$, with $|f^i| = k_i + 1$, $|e| = l + 1$. From the proof of Prop. 3, we can conclude that for all $i = 1,\ldots,m$ there exists a uniquely determined edge g^j with $g^j \in \mathscr{G}_i$. W.l.o.g., let $g^i := f^i_1$ for all $i = 1,\ldots,m$. Moreover, set $e^i_0 := e$ and $f^i_0 := f^i$ for all $i = 1,\ldots,m$. By the definition of a grid, we have $e = \bigcup_{s=0}^{l}(f^i_s \cap e) = \bigcup_{s=0}^{l}(f^j_s \cap e)$ for all $i,j = 1,\ldots,m$. W.l.o.g., let $f^i_s \cap e = f^j_s \cap e$ for all $s = 0,\ldots,l$ and $i,j = 1,\ldots,m$.

The vertex set of $S_\psi(x)$ is $V(S_\psi(x)) = \bigcup_{i=1}^{m} f^i = \bigcup_{i=1}^{m}\bigcup_{r=1}^{k_i}(f^i \cap e^i_r)$. It will be convenient to relabel them in the following manner: We set $V(S_\psi(x)) \ni v := x^i_{r_i}$ iff v is the uniquely determined vertex with $\{v\} = f^i \cap e^i_{r_i}$. Note that vertices with different labels are not necessarily distinct. Analogously, we assign labels to vertices w of H as follows: $w := y^i_{r_i}$ iff w is the uniquely determined vertex with $\{w\} = g^i_\cap e^i_{r_i}$. Since $y \in g^i$ for all $i = 1\ldots,m$, it follows $y^i_{r_i} \in V(S_\psi(y))$ for all $r_i = 0,\ldots,k_i$, $i = 1,\ldots,m$.

With this notation, we have to prove that the map defined by

$$x^i_{r_i} \mapsto y^i_{r_i} \tag{1}$$

for all $r_i = 0,\ldots,k_i$, $i = 1,\ldots,m$, is an isomorphism between $S_\psi(x)$and $S_\psi(y)$. Thus, we have to show that $x^i_{r_i} = x^j_{s_j}$ if and only if $y^i_{r_i} = y^j_{s_j}$. If $i = j$ this immediately implies $r_i = s_i$, otherwise $e^i_{r_i} \cap e^i_{s_i} \neq \emptyset$. Now assume that $i \neq j$. Suppose first $x^i_{r_i} = x^j_{s_j}$, i.e., $f^i \cap e^i_{r_i} = f^j \cap e^j_{s_j}$. If $y^i_{r_i} \neq y^j_{s_j}$, then $g^i \cap e^i_{r_i} \neq g^j \cap e^j_{s_j}$, which implies $e^i_{r_i} \neq e^j_{s_j}$ because otherwise g^j would be a diagonal of the grid $\mathscr{G}_i$. But the we find a 4-cycle $(x^i_{r_i}, e^i_{r_i}, y^i_{r_i}, g^i, y, g^j, y^j_{s_j}, e^j_{r_j})$ in H, with $g^i, g^j \in \psi$ and $e^i_{r_i}, e^j_{s_j} \in \varphi$, which contradicts the strong grid property. Thus, $x^i_{r_i} = x^j_{s_j}$ implies $y^i_{r_i} = y^j_{s_j}$, i.e., the mapping $x^i_{r_i} \mapsto y^i_{r_i}$ is well defined.

From analogous considerations we can conclude $x^i_{r_i} = x^j_{s_j}$ if $y^i_{r_i} = y^j_{s_j}$, which proves injectivity. Furthermore, $V(S_\psi(y)) = \bigcup_{i=1}^{m} g^i = \bigcup_{i=1}^{m}\bigcup_{r=1}^{k_i}(g^i \cap e^i_r) = \bigcup_{i=1}^{m}\bigcup_{r=1}^{k_i}\{y^i_r\}$. Thus, this mapping is surjective and therefore bijective. Moreover, since the edges of $S_\psi(x)$ and $S_\psi(y)$ are given by $f^i = \bigcup_{r=0}^{l}(f^i \cap e^i_r) = \bigcup_{r=0}^{l}\{x^i_r\}$ and $g^i = \bigcup_{r=0}^{l}(g^i \cap e^i_r) = \bigcup_{r=0}^{l}\{y^i_r\}$, resp., this mapping is an isomorphism. □

The isomorphism $S_\psi(x) \cong S_\psi(y)$ given in Equation (1) is *induced* by the edge $e \in \varphi$ in the sense that vertices of $S_\psi(x)$ are mapped onto vertices of $S_\psi(y)$ if and only if they are in the same edge that is parallel to e. The grid property is by itself not sufficient to determine these local isomorphism, as Fig. 1 shows: The vertices 2 and 6 are connected by a dashed edge, but the stars generated by the solid edges centered in 2 and 6, respectively, are not isomorphic.

3 Quotient hypergraphs

In this section, we prove that several results that have been shown for graphs in [4] are also true for hypergraphs.

Definition 6 (Quotient Hypergraph)**.** *Let $H = (V,E)$ be a hypergraph and let $\mathscr{P} = \{V_1,\ldots,V_k\}$ be a partition of the vertex set V of H. The* quotient hypergraph $H/\mathscr{P}$ *has vertex set $V(H/\mathscr{P}) = \{V_1,\ldots,V_k\}$ and $f = \{V_{i_1},\ldots;V_{i_r}\} \subseteq V(H/\mathscr{P})$ is an edge in $H/\mathscr{P}$ iff there exists an edge $e \in E(H)$ such that*

(i) $e \cap V_{i_j} \neq \emptyset$ for all $j = 1,\ldots,r$ and

(ii) $e \subseteq \bigcup_{j=1}^{r} V_{i_j}$.

By construction, the set

$$\mathscr{P}_{\varphi}^{R} := \left\{V(H_{\varphi}^{x}) \mid x \in V(H)\right\}$$

is a partition of $V(H)$ for every $\varphi \sqsubseteq R$. The quotient hypergraph $H/\mathscr{P}_{\varphi}^{R}$ has as its vertex sets the connected components H_{φ}^{x}. The set $\{H_{\varphi}^{x_1},\ldots,H_{\varphi}^{x_k}\}$ is an edge iff there are edges $e \in E(H)$ such that $e \cap V(H_{\varphi}^{w}) \neq \emptyset$ if and only if $w \in V(H_{\varphi}^{x_i})$ for $i \in \{1,\ldots,k\}$.

In the following we will be interested in particular in the complements of R-classes, i.e., in $\overline{\varphi} := E \setminus \varphi$. The corresponding partial hypergraphs are denoted by $H_{\overline{\varphi}}$, with connected components $H_{\overline{\varphi}}^{x}$ for a given $x \in V(H)$. We observe that $y \in V(H_{\overline{\varphi}}^{x})$ if and only if there is a path $P := (x = x_0, e_1, x_1, \ldots e_k, x_k = y)$ from x to y such that $e_i \notin \varphi$ for all $1 \le i \le k$.

Just like $\mathscr{P}_{\varphi}^{R}$, the sets

$$\mathscr{P}_{\overline{\varphi}}^{R} := \left\{V(H_{\overline{\varphi}}^{x}) \mid x \in V(H)\right\} \tag{2}$$

form a partition of $V(H)$ for every $\varphi \sqsubseteq R$. To see this, we note that $x \in V(H_{\overline{\varphi}}^{x})$ holds for all $x \in V(H)$. Thus, $P \neq \emptyset$ for all $P \in \mathscr{P}_{\overline{\varphi}}^{R}$ and $\bigcup_{P \in \mathscr{P}_{\overline{\varphi}}^{R}} P = V(H)$. Furthermore, $V(H_{\overline{\varphi}}^{x}) \cap V(H_{\overline{\varphi}}^{y}) \neq \emptyset$ if and only if x and y are in same connected component w.r.t. $\overline{\varphi}$, i.e., if and only if $V(H_{\overline{\varphi}}^{x}) = V(H_{\overline{\varphi}}^{y})$.

We furthermore will need the intersections

$$V_R(x) := \bigcap_{\varphi \sqsubseteq R} V(H_{\overline{\varphi}}^{x}).$$

These sets form the classes of the common refinement of the partitions $\mathscr{P}_{\overline{\varphi}}^{R}$, $\varphi \sqsubseteq R$, i.e.,

$$\mathscr{P}^{R} := \left\{ \bigcap_{\varphi \sqsubseteq R} V(H_{\overline{\varphi}}(x)) \mid x \in V(H) \right\} = \{V_R(x) \mid x \in V(H)\} \tag{3}$$

is again a partition of $V(H)$.

The main statement of this section is the following factorization theorem for quotient hypergraph $H/\mathscr{P}^{R}$. It directly generalizes the corresponding result for simple graphs shown in [4].

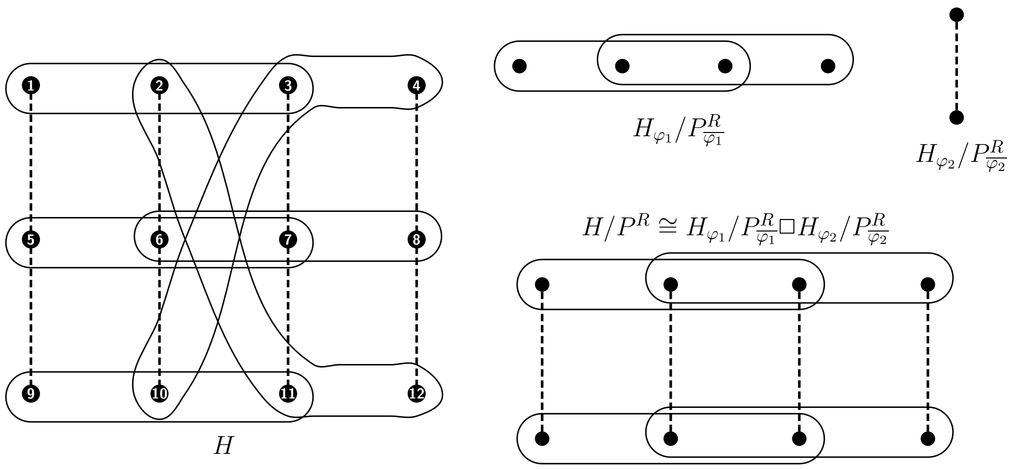

Figure 3: The equivalence relation R on $E(H)$ with equivalence classes φ_1 (solid), φ_2 (dashed) has the grid property. We have $\mathscr{P}^R_{\varphi_1} = \{\{1,5,9\},\{2,6,10\},\{3,7,11\},\{4,8,12\}\}$, $\mathscr{P}^R_{\varphi_2} = \{\{1,\ldots,4,9,\ldots,12\},\{5,\ldots,8\}\}$, and $\mathscr{P}^R = \{\{1,9\},\{2,10\},\{3,11\},\{4,12\}$. The corresponding quotient graphs $H_{\varphi_i}/\mathscr{P}^R_{\varphi_i}$, $i = 1,2$ and the product graph $H/\mathscr{P}^R$ are shown on the right-hand side.

Theorem 1. *If R is an equivalence relation with the grid property on $E(H)$ then*

$$H/\mathscr{P}^R \cong \Box_{\varphi \sqsubseteq R} H_\varphi/\mathscr{P}^R_\varphi.$$

To prove this theorem, we first have to verify the following lemma.

Lemma 6. *Let R be an equivalence relation on the edge set of a connected hypergraph H that satisfies the grid property and let $\varphi \sqsubseteq R$, $v \in V(H)$. Then for all $x \in V(H^v_\varphi)$ and all edges $e \in \psi \neq \varphi$ with $v \in e$, $\psi \sqsubseteq R$, there exists an edge e_x with $x \in e_x$ such that holds*

$$e \cap V(H^w_\varphi) = \emptyset \quad \text{if and only if} \quad e_x \cap V(H^w_\varphi) = \emptyset$$

for all $w \in V(H)$.

Proof. Let $v \in e := e_0 \in \psi$. For any $x \in V(H^v)$ there exists a path $P_{vx} := (v = v_0, f_1, v_1, \ldots, v_{k-1}, f_k, v_k = x)$ such that $f_i \in \varphi$ for all $i = 1,\ldots,k$. By the grid property, e_0 and f_1 span an $|e_0| \times |f_1|$-grid $\mathscr{G}_1$. Thus, there exists an edge $e_1 \in \mathscr{G}_1$, with $|e_1| = |e_0|$ such that $v_1 \in e_1 \in \psi$. Furthermore, for all $w \in e_0$ there is an edge $f^w_1 \in \varphi$ with $f^w_1 \cap e_1 \neq \emptyset$ if and only if $w \in f^w_1$. Hence, $e_0 \cap V(H^w_\varphi) = \emptyset$ if and only if $e_1 \cap V(H^w_\varphi) = \emptyset$. Moreover, since $v_1 \in f_2 \cap e_1$, $f_2 \in \varphi$, $e_1 \in \psi$, these two edges again span an $|e_1| \times |f_2|$-grid $\mathscr{G}_2$.

Inductively, we construct a collection of edges $e_1, \ldots, e_k$ such that e_{i-1} and f_i span an $|e_{i-1}| \times |f_i|$-grid $\mathscr{G}_i$ such that $e_i \in \mathscr{G}_i$ and $v_i \in e_i \cap f_i$. Therefore, by the same argument as before, we have $e_{i-1} \cap V(H^w_\varphi) = \emptyset$ if and only if $e_i \cap V(H^w_\varphi) = \emptyset$ and consequently $e_0 \cap V(H^w_\varphi) = \emptyset$ if and only if $e_i \cap V(H^w_\varphi) = \emptyset$ for all $i = 1, \ldots, k$. By setting $e_x := e_k$, the assertion follows. □

Proof of Thm 1. Let $\varphi_1, \ldots \varphi_n$ denote the equivalence classes of R. Let $x, v_1, \ldots, v_n \in V(H)$, where the v_i are not necessarily distinct. If $x \in V(H^{v_i}_{\varphi_i})$ for all $i = 1, \ldots n$ then $V_R(x) = \bigcap_{i=1}^n V(H^{v_i}_{\varphi_i})$.

We observe that for $1 \leq i \leq n$ the vertex set of $H_{\varphi_i}/\mathscr{P}^R_{\overline{\varphi}_i}$ is given by $V(H_{\varphi_i}/\mathscr{P}^R_{\overline{\varphi}_i}) = \{H^{v_i}_{\overline{\varphi}_i} \mid v_i \in V(H)\}$. Hence, we have

$$V\left(\Box_{i=1}^{n} H_{\varphi_i}/\mathscr{P}^R_{\overline{\varphi}_i}\right) = \left\{(H^{v_1}_{\overline{\varphi}_1},\ldots,H^{v_n}_{\overline{\varphi}_n}) \mid v_i \in V(H), i = 1,\ldots,n\right\},$$

where $(H^{v_1}_{\overline{\varphi}_1},\ldots,H^{v_n}_{\overline{\varphi}_n}) = (H^{u_1}_{\overline{\varphi}_1},\ldots,H^{u_n}_{\overline{\varphi}_n})$ if and only if $u_i \in V(H^{v_i}_{\overline{\varphi}_i})$ for all $i = 1,\ldots,n$.

We define a mapping $V(H/\mathscr{P}^R) \to V(\Box_{i=1}^{n} H_{\varphi_i}/\mathscr{P}^R_{\overline{\varphi}_i})$ as follows:

$$V_R(x) \mapsto (H^{v_1}_{\overline{\varphi}_1},\ldots,H^{v_n}_{\overline{\varphi}_n})$$

iff $x \in V(H^{v_i}_{\overline{\varphi}_i})$ for all $i = 1,\ldots n$.

For all $x \in V(H)$ there exist v_i, $i = 1,\ldots,n$, such that $x \in V(H^{v_i}_{\overline{\varphi}_i})$, e.g. choose $v_i = x$. And since from $x \in V(H^{v_i}_{\overline{\varphi}_i})$ and $x \in V(H^{u_i}_{\overline{\varphi}_i})$ follows $H^{v_i}_{\overline{\varphi}_i} = H^{u_i}_{\overline{\varphi}_i}$, this mapping is well defined.

Since both $x \in V(H^{v_i}_{\overline{\varphi}_i})$ and $y \in V(H^{v_i}_{\overline{\varphi}_i})$ we conclude $H^{x}_{\overline{\varphi}_i} = H^{y}_{\overline{\varphi}_i}$ we can conclude that this mapping is injective. To prove surjectivity, it suffices to show that $\bigcap_{i=1}^{n} V(H^{v_i}_{\overline{\varphi}_i}) \neq \emptyset$ for arbitrary $v_i \in V(H)$. We show by induction for all $k \leq n$ holds $\bigcap_{i=1}^{k} V(H^{v_i}_{\overline{\varphi}_i}) \neq \emptyset$. For $k = 1$ this is trivially fulfilled. Let $k \geq 1$ and suppose $\bigcap_{i=1}^{k} V(H^{v_i}_{\overline{\varphi}_i}) \neq \emptyset$. We have to show, that this implies $\bigcap_{i=1}^{k+1} V(H^{v_i}_{\overline{\varphi}_i}) \neq \emptyset$. From the induction hypothesis, we can conclude there must be a vertex $x \in V(H)$ such that $x \in V(H^{v_i}_{\overline{\varphi}_i})$ for all $i = 1,\ldots,k$ and hence $\bigcap_{i=1}^{k} V(H^{v_i}_{\overline{\varphi}_i}) = \bigcap_{i=1}^{k} V(H^{x}_{\overline{\varphi}_i})$ for all $i = 1,\ldots,k$. Therefore, we have to show

$$V(H^{x}_{\varphi_{k+1}}) \subseteq \bigcap_{i=1}^{k} V(H^{x}_{\overline{\varphi}_i}). \tag{4}$$

From that and Lemma 4 we obtain $\emptyset \neq V(H^{x}_{\varphi_{k+1}}) \cap V(H^{v_{k+1}}_{\overline{\varphi}_{k+1}}) \subseteq \bigcap_{i=1}^{k+1} V(H^{v_i}_{\overline{\varphi}_i})$ and the assumption follows.

Let $y \in V(H^{x}_{\varphi_{k+1}})$. Then there exists a path Q from x to y such that all edges of Q are in class φ_{k+1}. Clearly, they are not in class φ_i for $i = 1,\ldots k$ and therefore $y \in V(H^{x}_{\overline{\varphi}_i})$ for all $i = 1,\ldots k$, from what Equation (4) and finally surjectivity follows.

It remains to prove the isomorphism property, that is, we have to show that $\{V_R(x_1),\ldots,V_R(x_k)\}$ is an edge in $E(H/\mathscr{P}^R)$ if and only if $\{(H^{x_1}_{\overline{\varphi}_1},\ldots,H^{x_1}_{\overline{\varphi}_n}),\ldots,(H^{x_k}_{\overline{\varphi}_1},\ldots,H^{x_k}_{\overline{\varphi}_n})\}$ is an edge in $\Box_{i=1}^{n} H_{\varphi_i}/\mathscr{P}_{\overline{\varphi}_i}$.

Let $\{V_R(x_1),\ldots,V_R(x_k)\}$ be an edge in $E(H/\mathscr{P}^R)$. Thus, there exists edge $e \in E(H)$ such that $e \cap V_R(x_j) \neq \emptyset$ for all $j = 1,\ldots,k$ and $e \subseteq \bigcup_{j=1}^{k} V_R(x_j)$. Clearly, $e \in \varphi_m$ for some $m \in \{1,\ldots,n\}$, and hence $e \in \overline{\varphi}_l$ for all $l \neq m$, which implies $H^{x_1}_{\overline{\varphi}_l} = H^{x_j}_{\overline{\varphi}_l}$ for all $j = 1,\ldots,k$ and all $l \neq m$. We have to show that $\{H^{x_1}_{\overline{\varphi}_m},\ldots,H^{x_k}_{\overline{\varphi}_m}\}$ is an edge in $H_{\overline{\varphi}_m}/\mathscr{P}^R_{\overline{\varphi}_m}$. Recall, that $V_R(x_j) = \bigcap_{i=1}^{n} V(H^{x_j}_{\overline{\varphi}_i})$ and consequently, $e \cap V_r(x_j) \neq \emptyset$ implies $e \cap V(H^{x_j}_{\overline{\varphi}_m}) \neq \emptyset$, as well as $e \subseteq \bigcup_{j=1}^{k} V_R(x_j)$ implies $e \subseteq \bigcup_{j=1}^{k} V(H^{x_j}_{\overline{\varphi}_i})$. Thus, by definition of quotient hypergraphs we have $\{H^{x_1}_{\overline{\varphi}_m},\ldots,H^{x_k}_{\overline{\varphi}_m}\} \in E(H_{\overline{\varphi}_m}/\mathscr{P}^R_{\overline{\varphi}_m})$ and hence $\{(H^{x_1}_{\overline{\varphi}_1},\ldots,H^{x_1}_{\overline{\varphi}_n}),\ldots,(H^{x_k}_{\overline{\varphi}_1},\ldots,H^{x_k}_{\overline{\varphi}_n})\}$ is an edge in $\Box_{i=1}^{n} H_{\varphi_i}/\mathscr{P}_{\overline{\varphi}_i}$.

Now, let $\{(H^{x_1}_{\overline{\varphi}_1},\ldots,H^{x_1}_{\overline{\varphi}_n}),\ldots,(H^{x_k}_{\overline{\varphi}_1},\ldots,H^{x_k}_{\overline{\varphi}_n})\}$ be an edge in $\Box_{i=1}^{n} H_{\varphi_i}/\mathscr{P}_{\overline{\varphi}_i}$. Then there exists some $m \in \{1,\ldots,n\}$ such that $H^{x_1}_{\overline{\varphi}_l} = H^{x_j}_{\overline{\varphi}_l}$ for all $j = 1,\ldots,k$ and all $l \neq m$ and $\{H^{x_1}_{\overline{\varphi}_m},\ldots,H^{x_k}_{\overline{\varphi}_m}\} \in E(H_{\overline{\varphi}_m}/\mathscr{P}^R_{\overline{\varphi}_m})$. That is, there exists $e \in \varphi_m$ such that $e \cap V(H^{x_j}_{\overline{\varphi}_m}) \neq \emptyset$ for all $j=1,\dots,k$ and $e \subseteq \bigcup_{j=1}^{k} V(H_{\varphi_m}^{x_j})$. Hence, there exists $x \in e \cap H_{\varphi_m}^{x_1}$. By Lemma 6, we can conclude that there exists an edge $e' \in \varphi_m$ such that $x_1 \in e'$ and $e' \cap V(H_{\overline{\varphi}_m}^{x_j}) \neq \emptyset$ for all $j = 1,\dots,k$ and $e' \cap V(H_{\overline{\varphi}_m}^{w}) = \emptyset$ if $w \notin V(H_{\overline{\varphi}_m}^{x_j})$ for some $j \in \{1,\dots,k\}$. Let $z_j \in e' \cap V(H_{\overline{\varphi}_m}^{x_j})$. Consequently, $z_j \in V(H_{\overline{\varphi}_l}^{x_1}) = V(H_{\overline{\varphi}_l}^{x_j})$ for all $l \neq m$, hence $z_j \in V_R(x_j)$, and therefore $e' \cap V_R(x_j) \neq \emptyset$ for all $j=1,\dots,n$. Furthermore, since $e' \cap V(H_{\overline{\varphi}_m}^{w}) = \emptyset$ if $w \notin V(H_{\overline{\varphi}_m}^{x_j})$ for $j \in \{1,\dots,k\}$, we have $e' \subseteq \bigcup_{j=1}^{k} V(x_j)$ and consequently $\{V_R(x_1),\dots,V_R(x_k)\} \in E(H/\mathscr{P}^R)$, completing the proof. □

The following result provides a surprisingly simple characterization of product relations consisting of exactly two classes.

Theorem 2. *Let R be an equivalence relation on $E(H)$ consisting only of equivalence classes φ and $\overline{\varphi}$. Then $|V(H_{\varphi}^{x}) \cap V(H_{\overline{\varphi}}^{y})| = 1$ holds for all $x,y \in V(H)$ if and only if R is a product relation.*

The proof of Theorem 2 is essentially the same proof as that for the analogous results for simple graphs in [4]. In the graph case, these two theorems are intimately related to graph bundles [9], which intuitively can be seen as generalizations of products in the sense they consist of isomorphic fibres held together by a collection of squares. The grid property thus can be expected to play an important role for hypergraph bundles. We will explore this topic in detail in a forthcoming manuscript.

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Appendix: Basic Notation and Terminology

Hypergraphs. A *(finite) hypergraph* $H = (V,E)$ consists of a set of *vertices* $V = V(H)$ and a collection $E = E(H)$ of non-empty subsets of V known as *hyperedges*, or simply *edges*. Throughout this contribution, we consider only finite hypergraphs without multiple edges, i.e., E is a finite set. We write $m(H) = |E(H)|$ for the number of edges. A hypergraph $H = (V,E)$ is *simple* if no edge is contained in any other edge and $|e| \geq 2$ for all $e \in E$. A *simple graph* is a simple hypergraph such that $|e| = 2$ holds for all $e \in E$. Two vertices u and v are *adjacent* in $H = (V,E)$ if there is an edge $e \in E$ such that $u, v \in e$. Two edges $e, f \in E$ are adjacent if $e \cap f \neq \emptyset$. We say that a vertex v and an edge e of H are *incident* if $v \in e$.

A *loop* at $x \in V$ is an edge $\{x\} \in E$. We denote by $\mathcal{L}H$ the hypergraph obtained from H adding a loop at each vertex. Conversely, $\mathcal{N}H$ denotes the hypergraph obtained by removing all loops from H. A *partial hypergraph* $H' = (V',E')$ of a hypergraph $H = (V,E)$, denoted by $H' \subseteq H$, is a hypergraph such that $V' \subseteq V$ and $E' \subseteq E$. In the class of graphs partial hypergraphs are called *subgraphs*. The partial hypergraph $H[V'] = (V',E')$ is *induced* (by V' if $E' = \{e \in E \mid e \subseteq V'\}$. A partial hypergraph of a simple hypergraph is always simple. The *star* $S(v)$ with centre $v \in V$ is the partial hypergraph generated by the edges containing v.

A *walk* in $H = (V,E)$ is a sequence $P_{v_0,v_k} = (v_0, e_1, v_1, e_2, \ldots, e_k, v_k)$, where $e_1, \ldots, e_k \in E$ and $v_0, \ldots, v_k \in V$, such that each $v_{i-1} \neq v_i$ and $v_{i-1}, v_i \in e_i$ for all $i = 1, \ldots, k$. The walk P_{v_0,v_k} is said to *join* the vertices v_0 and v_k. A *path* is a walk where both the vertices $v_0, \ldots, v_k$ and the edges $e_1, \ldots, e_k$ are all distinct. A path between two edges e_i and e_j is path $P_{v_i v_j}$ joining vertices any pair of vertices $v_i \in e_i$ and $v_j \in e_j$. A *cycle* of length k, or k-cycle, is a sequence $(v_0, e_1, v_1, e_2, \ldots, v_{k-1}, e_k, v_0)$, such that $P_{v_0,v_{k-1}}$ is a path. The *star* $S_H(v)$ with centre $v \in V$ is the partial hypergraph of H generated by the edges containing v.

The *distance* $d_H(v,v')$ between two vertices v_0, v_k of H is the length of a shortest path joining them. We set $d_H(v,v') = \infty$ if there is no such path. A hypergraph $H = (V,E)$ is called *connected*, if any two vertices are joined by a (finite) path. A connected partial hypergraph $H' \subseteq H$ is called *convex*, if all shortest paths in H between two vertices in H' are also contained in H'. A not necessarily connected partial hypergraph $H' \subseteq H$ is convex, if all of its connected components are convex.

For two hypergraphs $H_1 = (V_1,E_1)$ and $H_2 = (V_2,E_2)$ a *homomorphism* from H_1 into H_2 is a mapping $\alpha : V_1 \to V_2$ such that $\alpha(e) = \{\alpha(v_1), \ldots, \alpha(v_r)\}$ is an edge in H_2 whenever $e = \{v_1, \ldots, v_r\}$ is an edge in H_1. A homomorphism from H_1 into H_2 implies also a mapping $\alpha_{\mathscr{E}} : E_1 \to E_2$. A mapping $\alpha : V_1 \to V_2$ is a *weak homomorphism* if edges are mapped either to edges or to vertices. A bijective homomorphism α is a hypergraph *isomorphism*

$\alpha(e) \in E_2$ if and only if $e \in E_1$. We say, H_1 and H_2 are *isomorphic*, in symbols $H_1 \cong H_2$, if there exists an isomorphism between them. An isomorphism from a hypergraph H onto itself is an *automorphism*.

The 2-section $[H]_2$ of a hypergraph $H = (V,E)$ is the graph (V,E') with $E' = \{\{x,y\} \subseteq V \mid x \neq y, \exists\, e \in E : \{x,y\} \subseteq e\}$, that is, two vertices are adjacent in $[H]_2$ if they belong to the same hyperedge in H.

Relations. We will consider equivalence relations R on E, i.e., $R \subseteq E \times E$ such that (i) $(e,e) \in R$, (ii) $(e,f) \in R$ implies $(f,e) \in R$ and (iii) $(e,f) \in R$ and $(f,g) \in R$ implies $(e,g) \in R$. The equivalence classes of R will be denoted by Greek letters, $\varphi \subseteq E$. We will furthermore write $\varphi \sqsubseteq R$ for mean that φ is an equivalence class of R.

A relation Q is finer than a relation R while the relation R is coarser than Q if $(e,f) \in Q$ implies $(e,f) \in R$, i.e, $Q \subseteq R$. In other words, for each class ϑ of R there is a collection $\{\chi | \chi \subseteq \vartheta\}$ of Q-classes, whose union equals ϑ. Equivalently, for all $\varphi \sqsubseteq Q$ and $\psi \sqsubseteq R$ we have either $\varphi \subseteq \psi$ or $\varphi \cap \psi = \emptyset$.

The Cartesian Product. Let H_1 and H_2 be two Hypergraphs. The *Cartesian product* $H = H_1 \Box H_2$ has vertex set $V(H) = V(H_1) \times V(H_2)$, that is the Cartesian product of the vertex sets of the factors and the edge set

$$E(H) = \big\{\{x\} \times f : x \in V(H_1), f \in E(H_2)\big\} \cup \big\{e \times \{y\} : e \in E(H_1), y \in V(H_2)\big\}. \tag{5}$$

The Cartesian product is associative and commutative, thus the Cartesian product of arbitrarily many hypergraphs is well defined. Every connected hypergraph has unique representation as Cartesian product of prime hypergraphs [5].

The mapping $p_i : V(\Box_{i=1}^n H_i) \to V(H_i)$ defined by $p_i(v) = v_i$ for $v = (v_1, v_2, \ldots, v_n)$ is called *projection* onto the i-th factor of H. The induced partial hypergraph H_i^w of H with vertex set $V(H_i^w) = \{v \in V(H) \mid p_j(v) = w_j,$ for all $j \neq i\}$ is called *H_i-layer through w*. It is isomorphic to H_i.

An equivalence relation R on the edge set $E(H)$ of a Cartesian product $H = \Box_{i=1}^n H_i$ of (not necessarily prime) hypergraphs H_i is a *product relation* if $e\,R\,f$ holds if and only if there exists a $j \in \{1, \ldots, n\}$ such that $|p_j(e)| > 1$ and $|p_j(f)| > 1$.